Problems on General Relativity: 8

December 7, 2021

Problem 1. Consider the Schwarzschild metric tensor

$$g = -(1 - \frac{r_s}{r})dt^2 + \frac{dr^2}{1 - \frac{r_s}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1}$$

for $r > r_s$, and the Killing vector

$$T = \partial_t. \tag{2}$$

1) Replace the coordinate t by the Eddington-Finkelstein advanced coordinate v defined by

$$dt = dv - \frac{dr}{1 - \frac{r_S}{r}} \tag{3}$$

and write the metric in the coordinates (v, r, θ, ϕ).

2) Calculate the Killing vector T is the new coordinates

3) Consider the surface (horizon)

$$r = r_S. \tag{4}$$

Assuming that the vector field T is directed to the future, show that every vector

$$X = X^v\partial_v + X^r\partial_r + X^\theta\partial_\theta + X^\phi\partial_\phi$$

directed to the future and defined at a point of the horizon, satisfies

$$X^r \leq 0. \tag{5}$$

4) Infer from 3) that matter can move from the $r > r_S$ region to the $r < r_S$ region, but not vice versa.

5) Calculate an explicit expression for the function $v(t, r)$.

Problem 2. Consider the metric tensor

$$g = -(1 - \frac{r_s}{r})dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{6}$$

and a timelike geodesic curve

$$\tau \mapsto \gamma(\tau) = (v(\tau), r(\tau), \frac{\pi}{2}, \phi(\tau)) \tag{7}$$

where τ is the proper time.

Express $\dot{v}, \dot{r}, \dot{\phi}$ by the constants of motion

$$E := -g(\dot{\gamma}, \partial_v), \quad \text{and} \quad L := g(\dot{\gamma}, \partial_\phi). \tag{8}$$

How could you use that result?

Problem 3* optional. Calculate the Noether charge of the symmetry

$$T = \partial_t \tag{9}$$

for a 2-sphere

$$S_r : t, r = \text{const}$$

in the the Schwarzshild spacetime (1). In what way does the charge depend on the 2-sphere, or even on a 2-surface?

Hint: The Noether charge of a vector field ξ is given by the integral of the following 2-form

$$Q^{(\xi)} = -\frac{c^4}{16\pi G}\frac{1}{2}\eta_{abcd}\nabla^c\xi^d dx^a \wedge dx^b \tag{10}$$

where η is the volume 4-form. Check

$$dQ^{(T)} = ? \tag{11}$$